Abstract

This paper develops a model of households’ choices regarding child care and their policy preferences. Households differ in the level of female human capital that affect both earnings and the quality of child care at home. Child care at home incurs the costs of not only the foregone wages but also wage reductions when returning to the labor market. By sending children to child care centers one can avoid such wage losses at a price. Taking all these factors into account, households also choose the optimal timing of child birth and their preferred policy: the allocation of a given budget between subsidizing child care centers and supporting child care at home.

We show that households with relatively low levels of human capital prefer subsidies to home care and choose home care and that households with relatively high levels of human capital prefer subsidies to child care centers and choose child care centers. Interestingly, households in the middle may prefer either one of the policies but end up choosing the subsidized child care. We also characterize the the conditions under which either of the two policies are supported in equilibrium with majority voting. Higher level of median human capital, lower cost of child care centers, higher costs of career interruption tend to result in the market care regime that subsidizes child care centers with no subsidies to child care at home.

Keywords: Career break; Child birth; Child care; Female human capital; Majority voting.
List of Tables

List of Figures

1 Households’ consumption resources for each child care technology. . . 5
1 Introduction

Child birth and ensuing care work are main obstacles for married women to continue their career. The wage loss can be incurred by career break if she have a child, and it is larger for her with high level of human capital. According to Miller (2005), it is estimated that there is 10 percent increase in earnings per year of delay, and the postponement premium is largest for college-educated women and those in professional and managerial occupations. Researchers typically find family penalty of 10-15 percent of their income for women with children, as compared to women without children (Waldfogel, 1998a). However, the married women can mitigate the wage cost by delaying their motherhood or by using market care instead of home care.

In Korea, the married women with a child are subsidized by government either if she choose home care or if she choose market care. The child care subsidy program is widely spread in developed countries. Especially, most of OECD countries have various forms of supporting child care. However, the care regime is various across countries in the sense that some countries tend to in-kind subsidy whereas other countries tend to in-cash subsidy. The cross-country difference in care regime is also true in western Europe even if their child care policies have been based on family-friendly ones including non-pecuniary support such as parental leave, expansion of child care facilities, and flexible working until 1970s (McIntosh, 1986; Castles, 2003; McDonald, 2006).

There exists a small literature on public provision of day care center (Bergstrom and Blomquist, 1996). Bergstrom and Blomquist (1996) suggested a political economy model of public provision of day care. They show how public subsidy is chosen by broad political support in case of day care. However, theoretical framework which considers political support for child care regime between two alternatives with explicit consideration on labor market environment is relatively rare. We develops theoretical model which incorporates households’ decision on the birth and child care, and we analyze who support which care regime. Furthermore, we analyze which care regime prevails in equilibrium.

In section 2, we describe structure of the economy, and analyze individual choices regarding when to have a child and how to take care of the child. Specifically, in section 2.1, we analyze individual households’ choice of child-bearing time for each child care technology and derive their indirect utility functions. In section 2.2, we
analyze individual households’ choice of child care technology. In section 3, we derive political preferences of households and find political equilibrium. Furthermore, we specify the condition which determines political outcome. In section 4, we study the comparative analysis on each equilibrium. Especially, we focus on how equilibrium outcome is affected by labor market environment and distribution of female human capital.

2 The Economic Choices

In this section, we develop a model in which a household chooses when to have a child and how to take care of the child, i.e., the child care technology. These choices are linked to the labor supply decision. Two available child care technologies are home care and market care. With home care, the female labor supply is zero during a given period of time, which incurs the costs of wage penalty at the time of labor market reentry as well as the foregone wage loss. With market care, however, no such costs arise.

Consider an economy populated by heterogeneous female households whose mass is normalized to unity. Households are differentiated by the level of female human capital $h$ that is distributed by $F$ on the interval $[h_{\text{min}}, h_{\text{max}}]$ of $\mathbb{R}^{+}$ with a density $f$.

Each household is endowed with a lifetime of $T$, which may be used either for work or for child care at home. Each household derives her felicity from three components: private consumption $c$, child care quality $x$, and the pleasure from a child $T - t$ where $t$ is birth timing. Specifically, the household’s lifetime utility is represented as

$$u = c + \gamma (\ln x + \ln(T - t)), \quad (1)$$

$\gamma > 0$. For the given time constraint $T$, each household chooses birth timing $t \in [0, T]$, and child care technology $i \in \{H, M\}$ to maximize her lifetime utility. Note that $H$ denotes home care and $M$ denotes market care. The child care quality is dependent on the child care technology:

$$x_i = \begin{cases} 
\tau h & \text{if } i = H \\
\tau \alpha & \text{if } i = M
\end{cases} \quad (2)$$

where $\tau > 0$ is the fixed nurturing time. According to (2), the quality of home care
is proportional to the female human capital whereas the quality of market care $\tau_\alpha$ is a positive constant. The household resources available for consumption is endogenously determined by the joint decision: the choice of child care technology $i$ and the corresponding choice of birth timing $t_i$. Specifically,

$$c_i = \begin{cases} 
wh[t_H + \delta(T - t_H - \tau)] + b & \text{if } i = H \\
whT - p + m & \text{if } i = M 
\end{cases}$$

(3)

where $w > 0$ is a wage rate for market labor, $0 < \delta < 1$ is the ratio of the wage rate for those who return to the labor market after career break to that for those who keep staying in the labor market, and $p > 0$ is a price of market care. We assume that $w$, $\delta$, and $p$ are given. Child care subsidies for households with $i = H$ and $i = M$ are denoted by $b$ and $m$ respectively. According to (3), the career break is unavoidable when households choose home care. This leads to consumption loss due to career break wage penalty and career break itself. The consumption loss incurred by career break is larger for households who have higher level of human capital. On the contrary, there is no career break when she chooses market care at the cost of $p$. Figure 1 summarizes the households’ resources for consumption in different phases of lifetime according to the choice of child care technology.

From the previous description of the economy, we can summarize the household’s problem as follows:

$$\max_{i, t_i} u_i = c_i + \gamma(\ln x_i + \ln(T - t_i)) \quad s.t. \quad (2), (3).$$

(4)

Note that $t_i$ and $i$ are simultaneously chosen. However, we separate two choices deliberately in ensuing analysis for clarity.
2.1 Choice of Birth Timing

We first analyze the choice of birth timing given the choice of child care technology. Households who choose $i = H$ solve

$$
\max_{t_H} u_H(t_H, b; h) = wh[t_H + \delta(T - t_H - \tau)] + b + \gamma(\ln \tau h + \ln(T - t_H)) \quad (5)
$$

and households who choose $i = M$ solve

$$
\max_{t_M} u_M(t_M, m; h) = whT + m - p + \gamma(\ln \tau \alpha + \ln(T - t_M)) \quad (6)
$$

The solutions to (5) and (6) are $t_H = T - \frac{\gamma}{(1 - \delta)wT} \equiv t_H(\delta; h)$ and $t_M = 0$ respectively. We focus on the interior solution so that $0 < t_H(\delta; h) < T - \tau$ for all $h$. In other words, we assume that the support of the distribution $F$ lies within the interval $(\gamma(1 - \delta)wT, \gamma(1 - \delta)w\tau)$.

Clearly, $\frac{\partial t_H(\delta; h)}{\partial h} < 0$ and $\frac{\partial t_H(\delta; h)}{\partial \delta} > 0$. That is, households tend to defer child birth when they have high levels of human capital and when they face high levels of career break wage penalty.

By substituting $t_H = t_H(\delta; h)$ and $t_M = 0$ into (5) and (6) respectively, we obtain indirect utility functions $v_H(b; h) \equiv u_H(t_H(\delta; h), b; h)$ and $v_M(m; h) \equiv u_M(0, m; h)$. Specifically, their expressions are as follows:

$$
v_H(b; h) = wh(T - \delta \tau) - \gamma + b + \gamma \ln\left(\frac{\gamma \tau}{(1 - \delta)w}\right), \quad (7)
$$

$$
v_M(m; h) = whT + m - p + \gamma \ln(\tau \alpha T). \quad (8)
$$

Note that $v_H(b; h)$ and $v_M(m; h)$ are strictly increasing in $b$ and $m$ respectively, and both are strictly increasing in $h$.

2.2 Choice of Child Care Technology

We introduce an auxiliary function $z(b, m; h) \equiv v_H(b; h) - v_M(m; h)$ so that

$$
i = \begin{cases} H, & \text{if and only if } z(b, m; h) > 0 \\ M, & \text{if and only if } z(b, m; h) \leq 0 \end{cases} \quad (9)
$$

where $z(b, m; h) \equiv b - (m - p) - (\delta wh \tau + \gamma) + \gamma \ln\left(\frac{\gamma}{(1 - \delta)\alpha wT}\right)$. Note that $\frac{\partial z(b, m; h)}{\partial b} > 0$, and $\frac{\partial z(b, m; h)}{\partial m} > 0$.
\[
\frac{\partial z(b,m;h)}{\partial m} < 0, \quad \text{and} \quad \frac{\partial z(b,m;h)}{\partial h} < 0.
\]

By construction, \(z(b,m;h)\) measures net benefit of choosing home care relative to market care. Note that \(z(b,m;h)\) consists of three components: the difference in net subsidies \(b - (m - p)\), the wage cost of choosing home care \(\delta wh\tau + \gamma\), and the utility gain from child care at home relative to market care \(\gamma \ln \left( \frac{\gamma}{(1-\delta)awT} \right)\). The wage cost of choosing home care can be regarded as sum of two parts: the foregone wage \(wh\tau\) and the career break wage penalty \(\gamma - wh(1 - \delta)\tau\).

According to (9), households’ choice of child care technology hinges on the sign of \(z(b,m;h)\). The following lemma summarizes how the sign of \(z(b,m;h)\) is assigned by the level of \(h\).

**Lemma 1** There is a unique level of human capital \(\tilde{h}\) such that \(z(b,m;h) \gtrless 0\) as \(h \gtrless \tilde{h}\) where \(\tilde{h} = a(\delta)(b - m + p - \phi(\delta)) \equiv \tilde{h}(b,m),\)

\[
a(\delta) \equiv \frac{1}{w\delta\tau} \quad \text{and} \quad \phi(\delta) \equiv \gamma - \gamma \ln \left( \frac{\gamma}{(1-\delta)awT} \right).
\]

**Proof.** This is easily obtained by simple algebra on \(z(b,m;h)\). \(\blacksquare\)

From Lemma 1 and (9), we can encapsulate the household’s decision as follows: \(i = H\) if and only if \(h < \tilde{h}\) and \(i = M\) if and only if \(h \geq \tilde{h}\). That is, individual households with relatively low human capital choose home care and those with relatively high human capital choose market care for given policies. This is because the wage cost of choosing home care increases as the level of household’s human capital increases.

Let \(0 \leq \lambda \leq 1\) be a fraction of households who choose home care, i.e., \(i = H\). Then, it is clear that

\[
\lambda = F(\tilde{h}(b,m)).
\]

Hence, it is also clear that \(\lambda = 0\) if \(\tilde{h} \leq h_{\text{min}}\), \(\lambda = 1\) if \(\tilde{h} \geq h_{\text{max}}\), and \(0 < \lambda < 1\) if \(h_{\text{min}} < \tilde{h} < h_{\text{max}}\). Henceforth, we assume \(f = \frac{1}{2\sigma} \cdot I[\tilde{h} - \sigma, \tilde{h} + \sigma]\) for analytical tractability. Note that \(I[\cdot]\) is an indicator function.

### 3 Policy Preferences and Equilibrium

In this section, we will study political preferences of households and characterize the voting equilibrium. We assume that all households vote sincerely. We also assume,
for simplicity, that the public budget for child care program is fixed and given by $B$. Then the budgetary relation is written as

$$B = \lambda b + (1 - \lambda)m,$$

(11)

$B > 0$ is a constant.

For expository convenience, we introduce a triplet $\Psi \equiv (b, m, \lambda)$ to denote the politico-economic outcome, or the policy regime. Note that, due to the restrictions of (10) and (11), only one value out of the triplet may be chosen independently. We will focus on two types of corner equilibria, i.e., the equilibrium with $m = 0$ or $b = 0$. Let $\Psi^*$ be an equilibrium outcome. Households choose between two policy regimes:

$$\Psi^*_1 = (0, m^*_1, \lambda^*_1) \text{ and } \Psi^*_2 = (b^*_2, 0, \lambda^*_2)$$

where $\lambda^*_1 = 1 - \frac{B}{m^*_1}$ and $\lambda^*_2 = \frac{B}{b^*_2}$.

In each regime, households choose child care technology, which depends on the level of $h$ relative to $\tilde{h}$. The level of $\tilde{h}$ in each regime is easily obtained by substituting $m = 0$ or $b = 0$ into $\tilde{h}(b, m)$ in Lemma 1. Specifically, they are expressed as

$$\tilde{h}(0, m^*_1) = a(\delta)(-m^*_1 + p - \phi(\delta)) \equiv \tilde{h}_1,$$

(12)

$$\tilde{h}(b^*_2, 0) = a(\delta)(b^*_2 + p - \phi(\delta)) \equiv \tilde{h}_2.$$

(13)

Note that $\tilde{h}_2 > \tilde{h}_1$ since $a(\delta) > 0$. Given the choice of child care technology $i$, the households choose policy regime $\Psi$ so as to solve the following problem:

$$\max_{\Psi} \{ \max_i \{ v_H(0; h), v_M(m^*_1; h) \}, \max_i \{ v_H(b^*_2; h), v_M(0; h) \} \}.$$

Then, the households’ policy preference is summarized as in the following proposition.

**Proposition 1** There is a unique level of human capital $\hat{h} \in (\tilde{h}_1, \tilde{h}_2)$ such that the preferred policy regime is $\Psi^*_2$ if $h < \hat{h}$ and $\Psi^*_1$ if $h \geq \hat{h}$, where

$$\hat{h} \equiv a(\delta)(b^*_2 - m^*_1 + p - \phi(\delta)).$$

**Proof.** See the Appendix.

According to Proposition 1, households with relatively high human capital tend to
prefer market care regime $\Psi^*_1$ to home care regime $\Psi^*_2$. Note that $\hat{h}$ is different from $\bar{h}$ in that $b$ and $m$ are taken from different policy regimes whereas $\bar{h}$ is determined by given a particular policy regime.

Now we consider the political choice of the policy regime by majority voting. Since $F$ has a density $\frac{1}{2\sigma}$ on $[\bar{h} - \sigma, \bar{h} + \sigma]$ by assumption, the fraction of households choosing home care under the given policy regime is specified as follows:

$$\lambda = \frac{1}{2\sigma} \cdot (\bar{h}(b, m) - (\bar{h} - \sigma)).$$

The equilibrium policy regime is either $\Psi^*_1$ or $\Psi^*_2$ depending on the level of $F(\hat{h})$, which then depends upon the level of $p$ and the relative size of $\delta$ given $\bar{h}$. We define a new parameter $\theta \equiv \bar{h} - a(\delta)(p - \phi(\delta)) = \theta(p, \delta, \bar{h})$ to discern the equilibrium policy regime precisely. From Proposition 1, Lemma 1, and (14), the following is established.

**Proposition 2** The policy regime chosen by majority voting hinges on the sign of $\theta$. Specifically, the equilibrium policy regime is

1. $\Psi^*_1$ if $\theta \in \Theta_1 \equiv (0, -\sigma + 2\sqrt{\sigma^2 - 2\sigma a(\delta)B})$;
2. $\Psi^*_2$ if $\theta \in \Theta_2 \equiv (\sigma - 2\sqrt{\sigma^2 - 2\sigma a(\delta)B}, 0)$;
3. $\Psi^*_1$ or $\Psi^*_2$ if $\theta = 0$.

**Proof.** See the Appendix.

We assume $\sigma > \frac{8a(\delta)B}{3}$ for $\Theta_1$ and $\Theta_2$ not to be empty on $\mathbb{R}$. Now we characterize the majority voting equilibrium.

**Corollary 1** Each equilibrium policy regime is characterized as follows:

1. Under the market care regime $\Psi^*_1$,

$$b^*_1 = 0, \quad m^*_1 = \frac{B}{1 - \lambda^*_1} = \frac{-(\sigma + \theta) + \sqrt{(\sigma + \theta)^2 + 8\sigma a(\delta)B}}{2a(\delta)}$$

and $0 < \lambda^*_1 = F(\bar{h}_1) < F(\hat{h}) < F(\bar{h}) = \frac{1}{2}$.
(2) Under the home care regime $\Psi_2^*$, 

$$b_2^* = \frac{B}{\lambda_2^*} = \frac{-(\sigma - \theta) + \sqrt{\left(\sigma - \theta\right)^2 + 8\sigma a(\delta)B}}{2a(\delta)}, \quad m_2^* = 0$$

and $\frac{1}{2} = F(\bar{h}) < F(\hat{h}) < F(\tilde{h}_2) = \lambda_2^* < 1$.

Proof. See the Appendix.

Note that $m_1^*, b_2^*, \lambda_1^*$ and $\lambda_2^*$ are affected by the level of $\theta$ as well as its sign whereas the political choice of the policy regime solely depends on the sign of $\theta$. Since $\theta$ is a function of $p, \delta$, and $\bar{h}$, the equilibrium policy regime is affected by these parameters. We will discuss it further in the following section.

To describe households’ choice of child care technology and policy preference in each regime, we partition the whole set of households into four groups depending on the level of human capital: $S_1 \equiv [\bar{h} - \sigma, \tilde{h}_1)$, $S_2 \equiv [\tilde{h}_1, \hat{h})$, $S_3 \equiv [\hat{h}, \tilde{h}_2)$, and $S_4 \equiv [\tilde{h}_2, \bar{h} + \sigma]$. Note that $\tilde{h}_1 < \hat{h} < \tilde{h}_2$ is clear. For brevity, we define a pair $q_k \equiv (i_k, \Psi_k)$ where $k$ is the group index, $k = 1, \ldots, 4$. Then, we can characterize the households’ decision for each group using $q_k$.

**Corollary 2** Households’ choice of child care technology and policy preference is summarized as follows:

1. If $\theta \in \Theta_1$, then $q_1 = (H, \Psi_2^*)$, $q_2 = (M, \Psi_2^*)$, $q_3 = (M, \Psi_1^*)$, and $q_4 = (M, \Psi_1^*)$;

2. If $\theta \in \Theta_2$, then $q_1 = (H, \Psi_2^*)$, $q_2 = (H, \Psi_2^*)$, $q_3 = (H, \Psi_1^*)$, and $q_4 = (M, \Psi_1^*)$.

**Proof.** This directly follows from Lemma 1, Proposition 1, and Proposition 2. ■

From Proposition 1, policy preferences of households depend on their levels of human capital relative to $\hat{h}$, regardless of the policy outcome. Specifically, households in $S_1$ and $S_2$ always choose $\Psi_2^*$ and households in $S_3$ and $S_4$ always choose $\Psi_1^*$. On the other hand, as shown in Lemma 1, households’ choices of child care technology depend on levels of human capital relative to $\tilde{h}(b, m)$, which is associated with the policy outcome. Specifically, under the home care regime $\Psi_2^*$, households in $S_1$, $S_2$, and $S_3$ choose $i = H$ and households in $S_4$ choose $i = M$. Under the market care
regime $Ψ^*_1$, households in $S_2$, $S_3$, and $S_4$ choose $i = M$ and households in $S_1$ choose $i = H$. Interestingly, all households in $S_1$ and $S_4$ have dominant strategy on child care technology in the sense that their choice of child care technology is invariant to the policy regime. Specifically, all households in $S_1$ always prefer home care to market care since it is not sufficiently beneficial to participate in labor market due to their low level of human capital relative to the cost of market care. Similarly, all households in $S_4$ choose regardless of the policy regime. However, households in $S_2$ and $S_3$ choose their child care technology in accordance with the policy regime. This is because their level of human capital is not low or high enough such that they are willing to change their choice of child care technology depending on the policy regime even if it is not consistent with their own policy preference. All households in $S_2$ and $S_3$ choose a child care technology that is subsidized under the realized policy regime.

4 Comparative Analysis

In this section, we study the aggregate characteristics of an economy that affect the policy regime in equilibrium. We first analyze how the equilibrium policy regime swings in response to changes in the parameters that affect the sign of $\theta$. Recall that $\theta = \theta(p, \delta, \bar{h})$.

Proposition 3 The sign of $\theta$ primarily depends on the level of $p$ and the size of $\delta$ with given $\bar{h}$.

(1) If $p \leq \phi(0)$, then there exists a unique $\tilde{\delta} \in (0, 1)$ such that $\theta \geq 0$ as $\delta \leq \tilde{\delta}$ where $\tilde{\delta} = \delta(\bar{h})$ with $d\delta(\bar{h})/d\bar{h} > 0$;

(2) If $p > \phi(0)$, then $\theta < 0$ for all $\delta \in (0, 1)$.

Proof. See the Appendix.

If market care is not too expensive, i.e. $p \leq \phi(0)$, then which regime is supported in equilibrium depends on the level of $\delta$ given $\bar{h}$. If $\delta$ is relatively small, i.e. $\delta < \tilde{\delta}$, then career break wage penalty is so harsh that more than half of households prefer market care regime to home care regime avoiding career break. Especially, for sufficiently low
level of $\bar{h}$, home care regime prevails except for $\delta$ close enough to zero. If $\delta$ is relatively large, i.e. $\delta > \bar{\delta}$, then career break wage penalty is not so harsh that more than half of households prefer home care regime to market care regime. Especially, for sufficiently high level of $\bar{h}$, market care regime prevails except for $\delta$ close enough to one. On the other hand, if market care is too expensive, i.e., $p > \phi(0)$, then the home care regime is supported by the majority in equilibrium.

Now we analyze how the equilibrium policies and $\lambda^*$ are affected by $\sigma$, $B$, and $\bar{h}$. The following proposition summarizes this.

**Proposition 4** The comparative effects of $\sigma$, $B$, and $\bar{h}$ on the equilibrium $\Psi^*$ are as follows:

1. If $\theta \in \Theta_1$, then $\partial m_1^* / \partial \sigma > 0$, $\partial m_1^* / \partial B > 0$, $\partial m_1^* / \partial \bar{h} < 0$ and $\partial \lambda_1^* / \partial \sigma > 0$,
   $\partial \lambda_1^* / \partial B < 0$, $\partial \lambda_1^* / \partial \bar{h} < 0$;
2. If $\theta \in \Theta_2$, then $\partial b_2^* / \partial \sigma > 0$, $\partial b_2^* / \partial B > 0$, $\partial b_2^* / \partial \bar{h} > 0$ and $\partial \lambda_2^* / \partial \sigma < 0$,
   $\partial \lambda_2^* / \partial B > 0$, $\partial \lambda_2^* / \partial \bar{h} < 0$.

**Proof.** See the Appendix.

First, consider an equilibrium under the market care regime with $0 < \lambda_1^* < \frac{1}{2}$. A rise in $\sigma$ increases the fraction of households who choose home care whereas a rise in $\bar{h}$ increases the fraction of households who choose market care. Accordingly, $\lambda_1^*$ and $m_1^*$ increase as $\sigma$ rises whereas they decrease as $\bar{h}$ rises. A rise in $B$ leads to an increase in $m_1^*$ and a decrease in $\lambda_1^*$ since it increases benefit for market care. Second, consider an equilibrium under the home care regime with $\frac{1}{2} < \lambda_2^* < 1$. A rise in $\sigma$ or $\bar{h}$ increases the fraction of households who choose market care. Accordingly, $\lambda_2^*$ decreases and $b_2^*$ increases. However, a rise in $B$ increases both $b_2^*$ and $\lambda_2^*$ since it increases benefit for home care.

### 5 Conclusions

We have developed a model of households choices regarding child care and their policy preferences. Households differ in the level of female human capital that affect
both earnings and the quality of child care at home. Child care at home incurs the costs of not only the foregone wages but also wage reductions when returning to the labor market. By sending children to child care centers one can avoid such wage losses at a price. Taking all these factors into account, households also choose the optimal timing of child birth and their preferred policy: the allocation of a given budget between subsidizing child care centers and child care at home.

We have showed that households with relatively low levels of human capital prefer subsidies to home care and choose home care and that households with relatively high level of human capital prefer subsidies to child care centers and choose child care centers. Interestingly, households in the middle may prefer either one of the policies but end up choosing the subsidized child care. We also have characterized the the conditions under which either of the two policies are supported in equilibrium with majority voting. Higher level of median human capital, lower cost of child care centers, higher costs of career interruption tend to result in the market care regime that subsidizes child care centers with no subsidies to child care at home. In reality, the level of median human capital and the leniency on the career interruption are mostly positively related each other. Thus, the child care regime which prevails in practice is various across countries.
Appendix

Proof of Proposition 1.

Since \( v_M(m; h) \) is strictly increasing in \( m \) and \( v_H(b; h) \) is strictly increasing in \( b \), \( v_M(m^*_1; h) > v_M(0; h) \) and \( v_H(b^*_2; h) > v_H(0; h) \) are clear. If \( b = 0 \), then \( v_H(0; h) > v_M(m^*_1; h) \) as \( h \leq \tilde{h}_1 \). If \( m = 0 \), then \( v_H(b^*_2; h) > v_M(0; h) \) as \( h \leq \tilde{h}_2 \). Thus, the rank among indirect utility functions for each household is summarized as follows:

\[
  v_H(b^*_2; h) > v_H(0; h) > v_M(m^*_1; h) > v_M(0; h) \quad \text{if} \quad \tilde{h} - \sigma \leq h < \tilde{h}_1, \\
  v_M(m^*_1; h) > v_M(0; h) > v_H(b^*_2; h) > v_H(0; h) \quad \text{if} \quad \tilde{h}_2 \leq h \leq \tilde{h} + \sigma, \\
  v_M(m^*_1; h) > v_H(0; h), \quad v_H(b^*_2; h) > v_M(0; h) \quad \text{if} \quad \tilde{h}_1 \leq h < \tilde{h}_2. 
\]

From (15) and (16), it is deduced that the preferred policy regime is \( \Psi^*_2 \) if \( h < \tilde{h}_1 \) and \( \Psi^*_1 \) if \( h \geq \tilde{h}_2 \). Note that, however, the rank among indirect utility functions in (17) is not fully determined, and thus we need to compare \( v_M(m^*_1; h) \) and \( v_H(b^*_2; h) \). By subtracting \( v_M(m^*_1; h) \) from \( v_H(b^*_2; h) \), we get the following preference relation:

\[
  v_H(b^*_2; h) \ngeq v_M(m^*_1; h) \quad \text{as} \quad h \leq \hat{h} \equiv a(\delta)(b^*_2 - m^*_1 + p - \phi(\delta)).
\]

Since \( a(\delta) > 0, m^*_1 > 0 \), and \( b^*_2 > 0 \), it is clear that \( \tilde{h}_1 < \hat{h} < \tilde{h}_2 \). Thus, the preferred policy regime is \( \Psi^*_2 \) if \( \tilde{h}_1 \leq h < \hat{h} \) and \( \Psi^*_1 \) if \( \hat{h} \leq h < \tilde{h}_2 \).

Proof of Proposition 2.

For \( \Psi^*_1 \) to be the equilibrium policy regime, it must be such that \( F(\hat{h}) < \frac{1}{2} \), or \( \hat{b} < \tilde{b} \). On the other hand, for \( \Psi^*_2 \) to be the equilibrium policy regime, it must be such that \( F(\hat{h}) > \frac{1}{2} \), or \( \hat{b} > \tilde{b} \). To find \( \hat{h} \), we need to specify equilibrium policy in each regime.

First, suppose that the equilibrium policy regime is \( \Psi^*_1 \). By solving budgetary relation (11) and equilibrium condition (14) for \( m \), we derive a quadratic equation of \( m \) as follows:

\[
  X(m) \equiv a(\delta)m^2 + (\sigma + \theta)m - 2\sigma B = 0. 
\]
Since \( X(0) < 0 \), the equation (18) has distinct real roots with opposite signs. Thus, \( m_1^* > 0 \) satisfies \( X(m_1^*) = 0 \) where

\[
m_1^* = \frac{-(\sigma + \theta) + \sqrt{(\sigma + \theta)^2 + 8\sigma \alpha(\delta)B}}{2\alpha(\delta)}. \tag{19}
\]

Second, suppose that the equilibrium policy regime is \( \Psi_2^* \). By solving (11) and (14) for \( b \), we derive a quadratic equation of \( b \) as follows:

\[
Y(b) \equiv a(\delta)b^2 + (\sigma - \theta)b - 2\sigma B = 0. \tag{20}
\]

Since \( Y(0) < 0 \), the equation (20) has distinct real roots with opposite signs. Thus, \( b_2^* > 0 \) satisfies \( Y(b_2^*) = 0 \) where

\[
b_2^* = \frac{-(\sigma - \theta) + \sqrt{(\sigma - \theta)^2 + 8\sigma \alpha(\delta)B}}{2\alpha(\delta)}. \tag{21}
\]

By substituting (19) and (21) into \( \hat{h} \) in Proposition 1, we obtain \( \hat{h} \):

\[
\hat{h} = \bar{h} + \sqrt{(\sigma - \theta)^2 + 8\sigma \alpha(\delta)B} - \sqrt{(\sigma + \theta)^2 + 8\sigma \alpha(\delta)B}. \tag{22}
\]

Then, the sign of \( \hat{h} - \bar{h} \) is determined such that

\[
\hat{h} - \bar{h} \lesssim 0 \quad \text{as} \quad \sqrt{(\sigma - \theta)^2 + 8\sigma \alpha(\delta)B} - \sqrt{(\sigma + \theta)^2 + 8\sigma \alpha(\delta)B} \gtrsim 0.
\]

Clearly, this is equivalent to the following:

\[
\hat{h} \lesssim \bar{h} \quad \text{as} \quad \theta \gtrsim 0. \tag{22}
\]

Now we find conditions for \( \Psi_1^* \) and \( \Psi_2^* \) to be stable. By substituting (14) into the budgetary relation (11), we obtain the following:

\[
B = \frac{1}{2}(b + m) - \frac{1}{2\sigma}(b - m)(\bar{h} - \hat{h}). \tag{23}
\]

By taking total differentiation on (23), we obtain the following:

\[
\frac{dB}{dt} = \frac{1}{2}(db + dm) - \frac{1}{2\sigma}(db - dm)(\bar{h} - \hat{h}) + \frac{1}{2\sigma}(b - m)d\hat{h} = 0. \tag{24}
\]
By total differentiation on $\tilde{h}$ in Lemma 1, we get the following:

$$d\tilde{h} = a(\delta)(db - dm)$$  \hspace{1cm} (25)

By substituting (25) into (24), we obtain the following:

$$dB = \frac{db}{2\sigma}(\sigma - \theta + 2a(\delta)(b - m)) + \frac{dm}{2\sigma}(\sigma + \theta - 2a(\delta)(b - m)) = 0.$$  \hspace{1cm} (26)

First, we identify stability condition for $\Psi^*_1$. By substituting $b = 0$ and $m = m^*_1$ in (19) into (26), we can simplify (26) as follows:

$$dB = \frac{db}{2\sigma}\left(2\sigma - \sqrt{(\sigma + \theta)^2 + 8\sigma a(\delta)B}\right) + \frac{dm}{2\sigma}\sqrt{(\sigma + \theta)^2 + 8\sigma a(\delta)B} = 0.$$  \hspace{1cm} (27)

If $2\sigma > \sqrt{(\sigma + \theta)^2 + 8\sigma a(\delta)B}$, then $\frac{db}{dm} < 0$ in (27). That is, $\Psi^*_1$ is stable if the following is satisfied:

$$\theta < -\sigma + 2\sqrt{\sigma^2 - 2\sigma a(\delta)B}.$$  \hspace{1cm} (28)

Second, we identify stability condition for $\Psi^*_2$. By substituting $m = 0$ and $b = b^*_2$ in (21) into (26), we can simplify (26) as follows:

$$dB = \frac{db}{2\sigma}\sqrt{(\sigma - \theta)^2 + 8\sigma a(\delta)B} + \frac{dm}{2\sigma}\left(2\sigma - \sqrt{(\sigma - \theta)^2 + 8\sigma a(\delta)B}\right) = 0.$$  \hspace{1cm} (29)

If $2\sigma > \sqrt{(\sigma - \theta)^2 + 8\sigma a(\delta)B}$, then $\frac{dm}{db} < 0$ in (29). That is, $\Psi^*_2$ is stable if the following is satisfied:

$$\theta > \sigma - 2\sqrt{\sigma^2 - 2\sigma a(\delta)B}.$$  \hspace{1cm} (30)

Therefore, by combining (28) with (22), we can establish that $\Psi^* = \Psi^*_1$ if

$$0 < \theta < -\sigma + 2\sqrt{\sigma^2 - 2\sigma a(\delta)B}$$

is satisfied. By combining (30) with (22), we can also establish that $\Psi^* = \Psi^*_2$ if

$$\sigma - 2\sqrt{\sigma^2 - 2\sigma a(\delta)B} < \theta < 0$$

is satisfied. If $\theta = 0$, then $\Psi^* = \Psi^*_1$ or $\Psi^* = \Psi^*_2$. 

\[\Box\]

16
Proof of Corollary 1.

As shown in (1) of Proposition 2, $\Psi_1^* = \Psi_1^1$ if $\theta \in \Theta_1$ and $m_1^*$ is given by (19). In this case, $\bar{h}_1 < \hat{h} < \bar{h}$, which implies that $\lambda_1^* = F(\bar{h}_1) < F(\hat{h}) < F(\bar{h}) = \frac{1}{2}$. Note that $\lambda_1^* = 0$ only if $X(B) \leq 0$, i.e., $\theta \geq \sigma - a(\delta)B$, which does not arise in $\Theta_1$ since $\sigma - a(\delta)B > -\sigma + 2\sqrt{\sigma^2 - 2\sigma a(\delta)B}$. Thus, $0 < \lambda_1^* = F(\bar{h}_1) < F(\hat{h}) < F(\bar{h}) = \frac{1}{2}$.

As shown in (2) of Proposition 2, $\Psi_2^* = \Psi_2^2$ if $\theta \in \Theta_2$ and $b_2^*$ is given by (21). In this case, $\bar{h}_2 > \hat{h} > \bar{h}$, which implies that $\lambda_2^* = F(\bar{h}_2) > F(\hat{h}) > F(\bar{h}) = \frac{1}{2}$. Note that $\lambda_2^* = 1$ only if $Y(B) \geq 0$, i.e., $\theta \leq a(\delta)B - \sigma$, which does not arise in $\Theta_2$ since $\sigma - 2\sqrt{\sigma^2 - 2\sigma a(\delta)B} > a(\delta)B - \sigma$. Thus, $\frac{1}{2} = F(\bar{h}) < F(\hat{h}) < F(\bar{h}_2) = \lambda_2^* < 1$. □

Proof of Proposition 3.

We first derive the properties of $\theta$ in the following lemma.

Lemma 2 The properties of $\theta(p, \delta, \bar{h})$ are as follows:

(1) $\lim_{\delta \to 1} \theta(p, \delta, \bar{h}) < 0$;

(2) If $p \leq \phi(0)$, then $\lim_{\delta \to 0} \theta(p, \delta, \bar{h}) > 0$ and $\partial \theta(p, \delta, \bar{h}) / \partial \delta < 0$ for all $\delta \in (0, 1)$;

(3) If $p > \phi(0)$, then $\lim_{\delta \to 0} \theta(p, \delta, \bar{h}) < 0$ and there is a unique $\delta_0 \in (0, 1)$ such that $\partial \theta(p, \delta, \bar{h}) / \partial \delta \geq 0$ as $\delta \leq \delta_0$ where $\phi(0) \equiv \gamma - \gamma \ln(\frac{\gamma}{\alpha \omega_T})$.

Proof.

The limit behaviors of $\theta(p, \delta, \bar{h})$ near $\delta = 0$ and $\delta = 1$ are clear. They are easily obtained by simple limit operation on $\theta(p, \delta, \bar{h})$. Thus, we focus on verifying the first order derivative property of $\theta(p, \delta, \bar{h})$.

By taking partial differentiation on $\theta(p, \delta, \bar{h})$ with respect to $\delta$, we obtain the following relation:

$$\frac{\partial \theta(p, \delta, \bar{h})}{\partial \delta} \geq 0 \quad \text{as} \quad p - \phi(0) \geq g(\delta) \equiv \gamma \left(\frac{\delta}{1 - \delta} + \ln(1 - \delta)\right).$$
Note that \( g'(\delta) > 0 \) for all \( \delta \in (0, 1) \) and \( \lim_{\delta \to 0} g(\delta) = 0 \). Moreover, it is readily known that \( \lim_{\delta \to 1} g(\delta) = \infty \) by L’Hôpital’s rule. Thus, if \( p > \phi(0) \), then there is a unique \( \delta_0 \in (0, 1) \) such that \( p - \phi(0) \geq g(\delta) \) as \( \delta \leq \delta_0 \) by intermediate value theorem. That is, \( \partial \theta(p, \delta, \bar{h}) / \partial \delta \geq 0 \) as \( \delta \leq \delta_0 \). If \( p \leq \phi(0) \), then \( p - \phi(0) < g(\delta) \) for all \( \delta \in (0, 1) \). That is, \( \partial \theta(p, \delta, \bar{h}) / \partial \delta < 0 \) for all \( \delta \in (0, 1) \).

We now prove Proposition 3. From (1) and (2) of Lemma 2, it is clear that there is a unique \( \bar{\delta} \in (0, 1) \) such that \( \theta \leq 0 \) as \( \delta \leq \bar{\delta} \) if \( p \leq \phi(0) \) by intermediate value theorem. Moreover, \( \bar{\delta} = \delta(\bar{h}) \) such that

\[
\frac{d\delta(\bar{h})}{d\bar{h}} = -\frac{\partial \theta(p, \bar{\delta}, \bar{h}) / \partial \bar{h}}{\partial \theta(p, \bar{\delta}, \bar{h}) / \partial \bar{\delta}}
\]

by implicit function theorem on \( \theta(p, \bar{\delta}, \bar{h}) = 0 \). Clearly, \( d\delta(\bar{h})/d\bar{h} > 0 \) since \( \partial \theta(p, \bar{\delta}, \bar{h}) / \partial \bar{h} > 0 \) and \( \partial \theta(p, \bar{\delta}, \bar{h}) / \partial \bar{\delta} < 0 \).

If \( p > \phi(0) \), then \( \partial \theta(p, \delta, \bar{h}) / \partial \delta \leq 0 \) as \( \delta \leq \bar{\delta} \) by (3) of Lemma 2. Note that \( \theta(p, \delta_0, \bar{h}) = \bar{h} - \gamma/(1 - \delta_0)w\tau < 0 \) by assumption. Since \( \theta(p, \delta_0, \bar{h}) = \max \theta(p, \delta, \bar{h}) < 0 \) given \( \bar{h} \), \( \theta(p, \delta, \bar{h}) < 0 \) for all \( \delta \in (0, 1) \).

Proof of Proposition 4.

To examine how \( m^*_1 \) and \( b^*_2 \) vary as \( \sigma, B, \bar{h} \) change, we use \( X(m) \) in (18) and \( Y(b) \) in (20) respectively. If \( \theta \in \Theta_1 \), then

\[
\frac{\partial X(m)}{\partial \sigma} < 0, \quad \frac{\partial X(m)}{\partial B} < 0, \quad \text{and} \quad \frac{\partial X(m)}{\partial \bar{h}} > 0
\]

are clear with \( \lambda^*_1 \in (0, 1/2) \) by simple calculus on \( X(m) \). Since \( m^*_1 \) is the solution to \( X(m) = 0 \), the following is deduced:

\[
\frac{\partial m^*_1}{\partial \sigma} > 0, \quad \frac{\partial m^*_1}{\partial B} > 0, \quad \text{and} \quad \frac{\partial m^*_1}{\partial \bar{h}} < 0.
\]

Since \( \lambda^*_1 = 1 - \frac{B}{m^*_1} \), the following is clear:

\[
\frac{\partial \lambda^*_1}{\partial \sigma} > 0 \quad \text{and} \quad \frac{\partial \lambda^*_1}{\partial \bar{h}} < 0.
\]
By simple calculus on (14), we obtain $\frac{\partial \lambda_1}{\partial B} = \frac{1}{2\sigma} \cdot \frac{\partial \tilde{h}}{\partial B}$ where $\frac{\partial \tilde{h}}{\partial B} = -a(\delta) \cdot \frac{\partial m}{\partial B} < 0$.

Thus, the following is clear:

$$\frac{\partial \lambda_1}{\partial B} < 0.$$ 

If $\theta \in \Theta_2$, then

$$\frac{\partial Y(b)}{\partial \sigma} < 0, \quad \frac{\partial Y(b)}{\partial B} < 0, \quad \text{and} \quad \frac{\partial Y(b)}{\partial \bar{h}} < 0$$

are clear with $\lambda_2^* \in (\frac{1}{2}, 1)$ by simple calculus on $Y(b)$. Since $b_2^*$ is the solution to $Y(b) = 0$, the following is deduced:

$$\frac{\partial b_2^*}{\partial \sigma} > 0, \quad \frac{\partial b_2^*}{\partial B} > 0, \quad \text{and} \quad \frac{\partial b_2^*}{\partial \bar{h}} > 0.$$ 

Since $\lambda_2^* = \frac{B}{b_2^*}$, the following is clear:

$$\frac{\partial \lambda_2^*}{\partial \sigma} < 0 \quad \text{and} \quad \frac{\partial \lambda_2^*}{\partial \bar{h}} < 0.$$ 

By simple calculus on (14), we obtain $\frac{\partial \lambda_2^*}{\partial B} = \frac{1}{2\sigma} \cdot \frac{\partial \tilde{h}_2^*}{\partial B}$ where $\frac{\partial \tilde{h}_2^*}{\partial B} = a(\delta) \cdot \frac{\partial b_2^*}{\partial B} > 0$. Thus, the following is clear:

$$\frac{\partial \lambda_2^*}{\partial B} > 0.$$ 

$\blacksquare$
References


